UNIVERSITY OF PENNSYLVANIA

CONCERNING CONTINUOUS CURVES AND CORRESPONDENCES

WILLIAM LEAKE AYRES

A THESIS

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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CONCERNING CONTINUOUS CURVES AND CORRESPONDENCES.*

By William Leake Ayres.

I. Introduction.

In this paper we shall discuss plane continuous curves† and especially continuous (1, 1) correspondences‡ between continuous curves. Part II contains a new definition of end-points and several theorems concerning continuous curves which are useful in the latter part of the paper. In part III we define and give several results concerning branch points and end-point homogeneity of continuous curves. In part IV we give a very general set of conditions under which a correspondence of subsets of two acyclic§ continuous curves can be extented to a continuous (1, 1) correspondence between the continuous curves themselves. The paper is confined to two dimensions but it is evident that a number of the theorems extend easily to n dimensions.

Finally I wish to thank Professor J. R. Kline, who suggested the problem of this paper and who has assisted materially in its preparation by many suggestions and criticisms.

II. Theorems on continuous curves.||

Theorem 1. *If M is a continuous curve and N is a subset of M which is also a continuous curve, then, for any positive number ε, $M-N$ contains at most a finite number of maximal connected subsets¶ of diameter greater than ε.***

Proof. Wilder†† has shown that $M-N$ consists of a set of maximal domains $[d]$, with respect to M, whose boundaries with respect to M belong

* Received January 11, 1927.

† For definitions and theorems relating to continuous curves see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs,* Bull. Amer. Math. Soc., vol. 29 (1923), pp. 289–302. Hereafter we shall refer to this paper as "*Report*".

‡ *Report,* p. 289.

§ A continuous curve which contains no simple closed curve is said to be *acyclic.*

|| Presented to the American Mathematical Society October 31, 1925.

¶ A connected set K is said to be a *maximal connected subset* of a set H if K is a subset of H and H contains no connected subset of which K is a proper subset. A subset N of a set M is said to be a *proper subset* if $M-N$ is not vacuous.

** The *diameter of a point-set* is the upper bound of the set of numbers $d(x, y)$, where x and y are points of the set and the symbol $d(x, y)$ denotes the distance from x to y.

†† R. L. Wilder, *Concerning continuous curves,* Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377, theorem 9.

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to N.* Suppose the set of maximal domains $[d]$ of $M-N$ of diameter greater than ε is infinite. If $\bar{d}$ denotes any domain of the set, $\bar{d}$ contains two points A and B such that $d(A, B) > \varepsilon$. There exists an arc AB from A to B every point of which lies in $\bar{d}$.† Since M is bounded and no two of the sets $[d]$ have common points, there exist two points $\bar{A}$ and $\bar{B}$ and two sequences $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ such that (1) the points $A_1, A_2, A_3, \cdots$ are distinct points of the set $[A]$ and $\bar{A}$ is the sequential limit point of this sequence, (2) the points $B_1, B_2, B_3, \cdots$ are distinct points of the set $[B]$ and $\bar{B}$ is the sequential limit point of this sequence, (3) for every i, the points A_i and B_i belong to the same domain of the set $[d]$.

It follows that $d(\bar{A}, \bar{B}) \geqq \varepsilon$. There exists an integer k such that if $i > k$, A_i and B_i lie within circles C_1 and C_2 of radii $\frac{1}{4}\varepsilon$ with centers at $\bar{A}$ and $\bar{B}$ respectively. Each arc A_iB_i $(i > k)$ contains a last point X_i in the order A_i to B_i in common with C_1 and the subarc X_iB_i of A_iB_i contains a first point Y_i in common with C_2. The sequence of arcs $X_{k+1}Y_{k+1}$, $X_{k+2}Y_{k+2}$, $X_{k+3}Y_{k+3}$, $\cdots$ contains a subsequence $\bar{X}_1\bar{Y}_1$, $\bar{X}_2\bar{Y}_2$, $\bar{X}_3\bar{Y}_3$, $\cdots$ having the following properties: (1) $\bar{X}_i$ is on C_1 and $\bar{Y}_i$ is on C_2 and every other point of $\bar{X}_i\bar{Y}_i$ lies outside both circles, (2) C_1 contains three points P_1, Q_1 and X such that every point $\bar{X}_i$ lies on the arc P_1Q_1X of C_1 and in the order $P_1\,Q_1\,\bar{X}_1\,\bar{X}_2\,\bar{X}_3\cdots X$, and C_2 contains three points P_2, Q_2 and Y such that every point $\bar{Y}_i$ lies on the arc P_2Q_2Y of C_2 and in the order $P_2\,Q_2\,\bar{Y}_1\,\bar{Y}_2\,\bar{Y}_3\cdots Y$, (3) the sequence $\bar{X}_1, \bar{X}_2, \bar{X}_3\cdots$ has the point X as its sequential limit point and the sequence $\bar{Y}_1, \bar{Y}_2, \bar{Y}_3, \cdots$ has as its sequential limit point the point Y, (4) each of the arcs $\bar{X}_i\bar{Y}_i$ lies in a maximal domain of $M-N$ of diameter greater than ε and no two lie in the same domain.

Each arc $\bar{X}_i\bar{Y}_i$ contains a point Z_i at a distance greater than or equal to $\frac{1}{2}\varepsilon$ from both $\bar{A}$ and $\bar{B}$. The set $[Z_i]$ contains a subset $Z_{n_1}, Z_{n_2}, Z_{n_3}, \cdots$ which has a sequential limit point Z. It follows that Z is at a distance greater than or equal to $\frac{1}{2}\varepsilon$ from both $\bar{A}$ and $\bar{B}$. Let C_3 denote a circle of radius $\varepsilon/8$ with center at Z. Since M is connected im kleinen there exists within C_3 a concentric circle C_4 such that every point of M lying within C_4 lies together with Z in a connected subset of M which lies

* A connected subset D of a continuous curve M is said to be a *domain* with respect to M, or an S-domain, if for every point P of D there exists a circle C with center at P such that the set of all points of M, which (1) lie interior to C and (2) lie with P in a connected subset of M that lies wholly interior to C, belong to D. The set of all limit points of D which do not belong to D constitutes the *boundary* of D. Cf. R. L. Wilder, loc. cit.

† For proof of this see R. L. Moore, *Concerning continuous curves in the plane,* Math. Zeit., vol. 15 (1922), pp. 254–261. See also R. L. Wilder, *loc. cit.*

entirely within C_3. Let Z_{m_1} be the first point of $[Z_{n_i}]$ which lies within C_4. The curve M contains a simple continuous arc α_1 from Z_{m_1} to Z which lies entirely within C_3.* The arc α_1 contains a last point T_1 in the order Z_{m_1} to Z in common with $\overline{X}_{m_1}\overline{Y}_{m_1}$ and the arc $T_1 Z$ of α_1 contains a first point T_2 in common with $\overline{X}_{m_1'}\overline{Y}_{m_1'}$.† Since C_3 is of radius $\varepsilon/8$ the arc $T_1 T_2$ of α_1 lies entirely within the simple closed curve J_1 formed by $\overline{X}_{m_1}\overline{Y}_{m_1}$, $\overline{X}_{m_1'}\overline{Y}_{m_1'}$, the arc $\overline{X}_{m_1}\overline{X}_{m_1'}$ of C_1 which lies on $P_1 Q_1 X$ and the arc $\overline{Y}_{m_1}\overline{Y}_{m_1'}$ of C_2 which lies on $P_2 Q_2 Y$. The arc $T_1 T_2$ contains in its interior a point V_1 of N, for otherwise the maximal domain of $M-N$ which contains $\overline{X}_{m_1}\overline{Y}_{m_1}$ would contain $\overline{X}_{m_1'}\overline{Y}_{m_1'}$ also which is contrary to the hypothesis that the arcs lie in distinct domains of the set $[d]$ of $M-N$.

Let C_5 denote a circle with center at Z lying within C_4, containing no point of $\overline{X}_{m_1}\overline{Y}_{m_1}$ and radius less than $\varepsilon/16$. Repeating the above argument with C_5 taking the place of C_3 we determine a point V_2 of N within the simple closed curve J_2 formed by $\overline{X}_{m_2}\overline{Y}_{m_2}$, $\overline{X}_{m_2'}\overline{Y}_{m_2'}$, the arc $\overline{X}_{m_2}\overline{X}_{m_2'}$ of C_1 which lies on $P_1 Q_1 X$ and the arc $\overline{Y}_{m_2}\overline{Y}_{m_2'}$ of C_2 which lies on $P_2 Q_2 Y$, where Z_{m_2} is the first point of the set $[Z_{n_i}]$ which lies in C_6 $(m_2 > m_1)$ and where C_6 is a circle concentric with C_5 such that every point of M within C_6 lies with Z in a connected subset of M lying wholly within C_5. Continue this process indefinitely each time taking the circle C_{2i+1} of radius less than $\varepsilon/2^{i+2}$ and such that no point of $\overline{X}_{m_{i-1}}\overline{Y}_{m_{i-1}}$ lies within it.

Thus there exists a set of points $V_1, V_2, V_3, \cdots$ such that for every value of i (1) V_i belongs to N and is at a distance from Z less than $\varepsilon/2^{i+2}$, (2) V_i lies interior to the simple closed curve J_i formed by $\overline{X}_{m_i}\overline{Y}_{m_i}$, $\overline{X}_{m_i'}\overline{Y}_{m_i'}$ the arc $\overline{X}_{m_i}\overline{X}_{m_i'}$ of C_1 which lies on $P_1 Q_1 X$ and the arc $\overline{Y}_{m_i}\overline{Y}_{m_i'}$ of C_2 which lies on $P_2 Q_2 Y$,

By property (1), Z is the sequential limit point of $V_1, V_2, V_3, \cdots$ and since N is closed, Z belongs to N. As N is connected im kleinen by hypothesis, there exists within the circle C_3 a circle $\overline{C}$ with center at Z such that every point of N within $\overline{C}$ can be joined to Z by an arc of N which lies wholly within C_3. By property (1), N contains a point V_j within $\overline{C}$. Let $\overline{\alpha}$ denote the arc of N from V_j to Z lying wholly within C_3. The point V_j is in the interior of the simple closed curve J_j and Z is in the exterior, thus $\overline{\alpha}$ must contain a point of J_j. Since the radius of C_3 is $\varepsilon/8$, $\overline{X}_{m_j}\overline{X}_{m_j'}$ and $\overline{Y}_{m_j}\overline{Y}_{m_j'}$ lie in the exterior of C_3 and thus can have no point in common with $\overline{\alpha}$. The arc $\overline{\alpha}$ can have no point in common with either $\overline{X}_{m_j}\overline{Y}_{m_j}$ or $\overline{X}_{m_j'}\overline{Y}_{m_j'}$ since these arcs belong to $M-N$. But these four arcs form J_j

* For a justification of the term "arc" here, see J. R. Kline, *Concerning approachability of simple closed and open curves*, Trans. Amer. Math. Soc., vol. 21 (1920), p. 453 and footnote.

† If m_i denotes n_j, then m_i' denotes n_{j+1}.

and therefore $\bar{\alpha}$ contains no point of J_j, which is a contradiction. This completes the proof of the theorem.

We may state the following more general form of this theorem. The proof, except for a few minor changes, is the same as for Theorem 1.

THEOREM 2. *If M, N_1, N_2, N_3, $\cdots$, N_k are continuous curves and $N_i (0 < i \leqq k)$ is a subset of M, then, for any positive number ε, $M-(N_1+N_2+\cdots+N_k)$ contains at most a finite number of maximal connected subsets of diameter greater than ε.*

Definition. A point P of a continuous curve M is said to be a *W-end-point,** or simply an *end-point,* of M if for every arc α of M joining P to any other point P' of M, it is true that the set $M-(\alpha-P)$ contains no connected subset consisting of more than one point which contains P.

THEOREM 3. *In order that a point P of a continuous curve M be an end-point of M, it is necessary and sufficient that* (1) *P belong to no simple closed curve which is a subset of M,* (2) *P be a non-cutpoint† of M.*

Proof. (a). *The conditions are necessary.* (1) Suppose P is an end-point of M and lies on a simple closed curve J of M. Let P' be any other point of J. Then J is the sum of two arcs from P to P',

$$J = PXP' + PYP'.$$

Let $PXP' = \alpha$. The set $M-(\alpha-P)$ contains a connected subset $PYP'-P'$ consisting of more than one point and containing P. Then P cannot be an end-point of M by definition, which is a contradiction.

(2) Suppose that P is an end-point of M which is also a cutpoint of M. Let P' be any other point of M and let α be an arc of M whose end-points are P and P'. As P is a cutpoint,

$$M-P = M_1 + M_2,$$

where M_1 and M_2 are non-vacuous mutually separated sets.‡ Now $\alpha-P$, being connected, must lie wholly in M_1 or M_2. Suppose it lies in M_1. By a theorem due to Wilder,§ M_2 is a set of domains with respect to M

* This definition was given by R. L. Wilder, *loc. cit.*, p. 358. Cf. also H. M. Gehman, *Concerning end points of continuous curves and other continua.*, Bull. Amer. Math. Soc., vol. 32 (1926), p. 218 (abstract).

† If M is a continuum and P is a point of M, then P is said to be a *non-cutpoint* or a *cutpoint* according as $M-P$ is connected or not. Cf. R. L. Moore, *Concerning the cut-points of continuous curves and of other closed and connected point-sets,* Proc. Nat. Acad., vol. 9 (1923), pp. 101–106.

‡ Two sets H and K are said to be *mutually exclusive* if they have no common points. They are said to be *mutually separated* if they are mutually exclusive and neither contains a limit point of the other. The symbol $H \cdot K$ is used to denote the set of common points of the sets H and K.

§ *Loc. cit.*, Theorem 9, p. 360.

whose boundary points with respect to M are contained in $M_1 + P$. Let D be one of these domains. As D is a subset of M_2, no point of M_1 can be a limit point of D and thus P must be the entire boundary of D. Then $D + P$ is connected, consists of more than one point and contains P, which is contrary to the assumption that P is an end-point.

(b). *The conditions are sufficient.* Let P be a point of M which satisfies the two conditions and let us suppose that P is not an end-point of M. Then there exists a point P' of M and an arc α of M whose end-points are P and P' such that $M-(\alpha-P)$ contains a connected set N consisting of more than one point and containing P. Since P is a non-cutpoint, $M-P$ is connected. Divide $M-P$ into two sets, $\alpha-P$ and $\bar{M}$. Then $\bar{M}$ contains $N-P$. Let q be any point of $N-P$ and let D_q be that maximal domain with respect to M which lies in $\bar{M}$ and contains q. As N is connected, one of the sets $N \cdot D_q$ and $N - N \cdot D_q$ must contain a limit point of the other. The set $N \cdot D_q$ cannot contain a limit point of $N - N \cdot D_q$ as no point of the domain D_q is a limit point of a set of points of M lying outside the domain. Since D_q is a maximal domain of $\bar{M}$ and $N-(N \cdot D_q + P)$ lies in $\bar{M}$, no point of $N-(N \cdot D_q + P)$ is a limit point of $N \cdot D_q$. Therefore P is a limit point of $N \cdot D_q$, and consequently of D_q. As D_q is a maximal domain of $\bar{M}$, D_q and $\bar{M} - D_q$ are mutually separated; and as $M-P$ is connected, $\alpha-P$ must contain a point x which is a limit point of D_q. There exist* arcs α' and α'', whose end-points are x and q and P and q respectively such that D_q contains $\alpha'-x$ and $\alpha''-P$.

There exists an arc β, whose end-points are x and P, every point of which belongs to $\alpha' + \alpha''$ and such that D_q contains $\beta-(P+x)$. The arc β and the arc Px of α have no points in common except their end-points and thus their sum is a simple closed curve which is a subset of M and contains P. But this is contrary to the hypothetis that P lies on no simple closed curve which is a subset of M.

THEOREM 4.† *If x and y are two distinct points of a continuous curve M, then no point of an arc xy of M, except possibly the points x and y, is an end-point of M.*

Proof. Let P denote any point of an arc xy of M distinct from x and from y. Let α denote the arc Px which is a subset of the arc xy. Then $M-(\alpha-P)$ contains a connected subset, namely the arc Py of xy, consisting of more than one point and containing P. Thus, by definition, P cannot be an end-point of M.

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* Cf. R. L. Wilder, *loc. cit.*, Theorem 1, p. 342.

† R. L. Wilder, *loc. cit.*, Theorem 14, p. 364, gives a proof of this theorem for the acyclic case. Cf. also R. L. Moore, *Concerning the cut-points etc.*, loc. cit., Theorem C.

THEOREM 5. *If K denotes the set of all end-points of a continuous curve M then K contains no connected subset consisting of more than one point.*

*Proof.** Suppose that K contains a connected subset $\bar{K}$ which contains more than one point. Let P be any point of $\bar{K}$. Let P' be any other point of M and α be an arc of M whose end-points are P and P'. Let P'' denote a point of $\alpha - (P + P')$. Then by Theorem 4, no point of the arc PP'' of α except possibly the point P is a point of K. Thus the set $M - (PP'' - P)$ contains the set K and consequently $\bar{K}$. But this is contrary to the assumption that P is an end-point of M.

THEOREM 6. *If K denotes the set of all end-points of a continuous curve M, then no subset of K disconnects M even in the weak sense.*†

Proof. Suppose K contains a subset $\bar{K}$ such that $M - \bar{K}$ is not connected in the strong sense. Then $M - \bar{K}$ contains two points A and B which lie in no closed and connected subset of $M - \bar{K}$. The curve M contains an arc AB whose end-points are A and B.‡ Since A and B lie in no closed and connected subset of $M - \bar{K}$, the arc AB must contain a point of $\bar{K}$, which we will call P, and P must be distinct from A and from B. The point P belongs to $\bar{K}$ and thus to K. But this is impossible by Theorem 4.

III. On branch points and homogeneity.§

Definition. We shall say that a continuous curve M is *homogeneous with respect to n end-points* if for each two sets of n distinct end-points of M, $A_1, A_2, A_3, \cdots, A_n$ and $B_1, B_2, B_3, \cdots, B_n$, there exists a continuous (1,1) transformation T|| of M into itself such that $T(A_i) = B_i$ for each

* For a proof of this theorem for the acyclic case, see R. L. Wilder, *loc. cit.*, Theorem 15, p. 365.

† A point-set is said to be *connected in the weak sense* if no matter how it is divided into two mutually exclusive subsets, one of these subsets contains a limit point of the other. The set M is said to be *connected in the strong sense* if every two points of M lie together in some closed subset of M which is connected in the weak sense. A point set K is said to *disconnect a connected point-set M in the weak sense* if $M - M \cdot K$ is not connected in the strong sense. For these definitions, see R. L. Moore, *Concerning the cut-points etc., loc. cit.;* Theorem F of Moore's paper together with Theorem 3 of this paper give a proof of the acyclic case of Theorem 6.

‡ Every two points of a continuous curve M can be joined by a simple continuous arc which is a subset of M. Cf. R. L. Moore, *A theorem concerning continuous curves,* Bull. Amer. Math. Soc., vol. 23 (1917), pp. 233–236. See also R. Tietze, *Über stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven,* Math. Zeit., vol. 5 (1919), pp. 284–291; S. Mazurkiewicz, *Sur les lignes de Jordan,* Fund. Math., vol. 1 (1920), pp. 166–209, and earlier papers referred to therein.

§ The results of this section, except Theorems 11 and 12, were presented to the Society January 2, 1926. Theorems 11 and 12 were presented May 1, 1926.

|| Cf. *Report*, p. 289. We shall use the terms "transformation" and "correspondence" interchangeably.

$0 < i \leqq n$. If a continuous curve is homogeneous with respect to n end-points for every positive integer n, we shall say that it is *homogeneous with respect to its end-points* or *end-point homogeneous.**

Definition. Suppose P is a point of a continuous curve M which lies on no simple closed curve of M. Then $M-P$ is a finite or at most countable† set of trees $[T]$.‡ If the set $[T]$ consists of just one tree, P is an end-point of M by Theorem 3. If the set $[T]$ consists of just two trees, we shall call P an *ordinary cut-point* of M. If the set $[T]$ consists of three or more trees, we shall call P a *branch point* of M. If the set $[T]$ consists of a finite number k $(k > 2)$ of trees we shall say that P is a branch point *of order* k. If the set $[T]$ is infinite, either there are only a finite number n of trees each of which contains just one end-point of M or there are an infinite number of such trees. In these two cases we shall say that P is a branch point *of order* $n\infty$ and $\infty\,\infty$ respectively. (Note: n may be zero.)

THEOREM 7. *Suppose* (1) *M is an acyclic continuous curve and A and B are end-points of M,* (2) *there exists a continuous* (1,1) *transformation T of M into itself such that $T(A) = B$ and $T(B) = A$, then the arc AB§ of M contains a point which remains invariant under T.*||

The proof of Theorem 7 affords no difficulty and will be omitted.

THEOREM 8. *If all, except possibly three, of the end-points of an acyclic continous curve M remain invariant under a continuous* (1,1) *transformation T of M into itself, then all of the branch points of M remain invariant under T.*

Proof. Under the hypothesis that not more than three end-points of M are affected by T three cases arise: (1) all of the end-points of M remain invariant under T, (2) all except two, E_1 and E_2, of the end-points of M remain invariant and $T(E_1) = E_2$, $T(E_2) = E_1$, (3) all except three,

* S. Mazurkiewicz, *Sur les continus homogènes*, Fund, Math., vol. 5 (1924), pp. 137–146, has defined a homogeneous set of points and has shown that a plane bounded continuous curve which is homogeneous is a simple closed curve. The above definition gives a somewhat similar notion which is applicable to acyclic continuous curves.

† Cf. Theorem 1. Cf. also R. L. Wilder, *loc. cit.*, Theorem 9.

‡ If N is a subcontinuum of a continuous curve M, a maximal connected subset of $M-N$ which has only one limit point in N is called a *tree* with respect to N or a tree in $M-N$. The limit point in N is called the *foot of the tree.* Cf. H. M. Gehman, *On extending a continuous* (1, 1) *correspondence of two plane continuous curves to a correspondence of their planes,* Trans. Amer. Math. Soc., vol. 28 (1926), pp. 252–265.

§ If A and B are two points of an acyclic continuous curve M then AB denotes the arc of M from A to B.

|| W. Scherrer, *Über ungeschlossene stetige Kurven*, Math. Zeit., vol. 24 (1925), pp. 125–130, proves that if T is *any* continuous (1,1) transformation of M into itself then M contains an invariant point. Then Theorem 7 states that under condition (2) one such point lies on the arc AB.

E_1, E_2 and E_3 remain invariant and $T(E_1) = E_2$, $T(E_2) = E_3$, $T(E_3) = E_1$. We shall show that in each of these cases all of the branch points must remain fixed under T.

In case (2) the arc E_1E_2 contains a point P which is invariant under T by Theorem 7. We shall now show that E_1E_2 cannot contain a branch point distinct from P and thus prove that the arc E_1E_2 contains not more than one branch point and if there exists one it remains invariant under T. Suppose the arc E_1E_2 contains a branch point B distinct from P. The point B must lie on either $\overset{\frown}{PE_1}$ or $\overset{\frown}{PE_2}$,* suppose $\overset{\frown}{PE_1}$. As B is a branch point there is at least one end-point E_3 such that the arc BE_3 has only the point B in common with E_1E_2. Since $T(PE_1) = PE_2$, B' (i. e. $T(B)$) must lie on $\overset{\frown}{PE_2}$. The point E_3 is invariant under T since we are considering the case in which only two end-points are affected by T. Then $T(E_3B) = E_3B'$ and, as the arc E_3B' contains P, the arc E_3B must contain a point Q such that $T(Q) = P$. Thus M contains two points, P and Q which have P as their correspondent under T, which is contradictory to the hypothesis that T is a (1,1) correspondence.

We shall show that in case (3) each of the three arcs, E_1E_2, E_2E_3, E_1E_3, contains but one branch point and this branch point is common to the three arcs and remains invariant under T. Let P be the first point of the arc E_1E_2 in the order from E_1 to E_2 which is a point of E_2E_3. The point P is distinct from E_1, E_2 and E_3 by Theorem 4. Then the subarc PE_3 of E_2E_3 has no point except P in common with the arc E_1E_2, P is the only point which belongs to all three of the arcs E_1E_2, E_2E_3, E_1E_3 and P is a branch point of M. We have now

$$T(E_1E_2) = E_2E_3, \qquad T(E_2E_3) = E_3E_1, \qquad T(E_1E_3) = E_2E_1.$$

Now P lies on each of the three arcs E_1E_2, E_2E_3, E_1E_3, and thus P' must lie on each of the three transformed arcs. But as P is the only point which lies on each of the three arcs E_2E_3, E_3E_1 E_2E_1, $P' = T(P) = P$. Thus P is invariant under T. Now suppose one of the three arcs from P, say PE_1, contains a branch point B distinct from P. The argument continues exactly as in case (2) above and we obtain a point Q such that $Q \neq P$ and $T(Q) = P$, which contradicts the hypothesis that T is a (1,1) transformation.

Now, in any one of the three cases, let us suppose that M contains a branch point B such that $T(B) = B' \neq B$. Let E_1E_2 be a maximal arc of M containing B and B';† then E_1 and E_2 are end-points of M.

* If AB is an arc, $\overset{\frown}{AB}$ denotes $AB-(A+B)$.

† S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fund. Math., vol. 2 (1921), pp. 119–130.

We suppose the order $E_1 B B' E_2$. We see easily that, as B is a branch point of M, B' is also and thus there exist two end-points E_3 and E_4 of M such that B and B' are the only points which the arcs $E_3 B$ and $E_4 B'$ have in common with the arc $E_1 E_2$. Not more than one of the four points E_i can be affected by T, for if two fail to remain fixed under T the arc joining these two must contain one of the two points B or B', neither of which remains invariant under T. But we proved above that if two end-points fail to remain fixed under T then the arc of M joining them can contain at most one branch point and this branch point must remain fixed under T. It follows then that not more than one of the four points E_i can be affected by T. And if one of the points E_i fails to remain fixed under T, E_i and E_i' lie in the same tree of $M-BB'$.

Then at least one of the two points E_2 and E_4, say E_2, and at least one of the points E_1 and E_3, say E_1, remain invariant under T. Since $T(E_1 B) = E_1 B'$ and $\underset{\frown}{E_1 B'}$ contains B, $\underset{\frown}{E_1 B}$ contains a point P_1 such that $T(P_1) = B$. Similarly $T(E_3 B) = E_3' B'$ and $\underset{\frown}{E_3' B'}$ contains B since E_3 and E_3', whether identical or not, lie in the same tree of $M-BB'$. Therefore $\underset{\frown}{E_3 B}$ contains a point P_2 such that $T(P_2) = B$. But $P_1 \neq P_2$ and B is the correspondent of each point under T, which is contrary to the hypothesis that T is a (1,1) correspondence.

THEOREM 9. *In order that an acyclic continuous curve M be homogeneous with respect to its end-points it is necessary and sufficient that M contain a point P such that M consists of a set of arcs from P to the end-points of M, no two of which have any point in common except P.*

Proof. The condition is obviously sufficient; we shall show that it is also necessary. Let E denote the set of all end-points of M and let e_1 and e_2 be points of E. The arc $e_1 e_2$ of M must contain a branch point b_1 for otherwise M consists of merely the arc $e_1 e_2$ and the theorem is true. The set $M - b_1$ consists of at least three trees, one M_{e_1} containing e_1, one M_{e_2} containing e_2, and at least one other M_{e_3} containing a point e_3 of E. We shall prove that $M_{e_1} = e_1 b_1 - b_1$.

Suppose that M_{e_1} contains a point p which is not on $e_1 b_1$. Let $e_4 b_1$ be a maximal arc of $M_{e_1} + b_1$ which contains p and b_1 and let b_2 be the first point which the arc $e_4 b_1$ has in common with $e_1 b_1$ in the order e_4 to b_1. Then e_4 is a point of E and b_2 is a branch point of M. By hypothesis there exists a continuous (1,1) transformation T of M into itself such that $T(e_1) = e_2$, $T(e_2) = e_1$, $T(e_3) = e_3$, $T(e_4) = e_4$. By Theorem 7 the arc $e_1 e_2$ contains a point q which corresponds to itself under T. Five cases may arise according as (1) q lies between e_1 and b_2, (2) $q = b_2$, (3) q lies between b_2 and b_1, (4) $q = b_1$, (5) q lies between b_1 and e_2.

Let us consider the first case. Since $e_1' = e_2$ and $e_2' = e_1$, $T(e_1 q) = e_2 q$ and $T(e_2 q) = e_1 q$. Then b_1' and b_2' lie on $e_1 q$. The set $e_3 b_1 + e_4 b_2$ is not connected while $T(e_3 b_1 + e_4 b_2) = e_3 b_1' + e_4 b_2'$ is connected since each component is an arc and they have the point q in common. But this is a contradiction since the connectivity of a set is invariant under a continuous transformation. In each of the other four cases we arrive at a contradiction in exactly the same way by showing that $e_3 b_1 + e_4 b_2$ is not connected while $e_3 b_1' + e_4 b_2'$ is connected. Thus $M_{e_1} = e_1 b_1 - b_1$.

Let M_e be any other tree of $M - b_1$, and let e be a point of E in M_e. Suppose M_e contains a point which is not a point of eb_1. Then M_e contains an end-point $\bar{e}$ distinct from e. We may arrive at a contradiction exactly as in the case of M_{e_1} above by considering the continuous (1,1) transformation $\bar{T}$ of M into itself such that

$$\bar{T}(e) = e_1, \qquad \bar{T}(e_1) = e, \qquad \bar{T}(e_4) = e_4, \qquad \bar{T}(\bar{e}) = \bar{e}.$$

Thus $M_e = eb_1 - b_1$. Therefore every tree of $M - b_1$ is a set consisting of an arc of M from b_1 to an end-point of M minus the point b_1, which proves the necessity of the condition.

As M is connected im kleinen we see that the set of arcs of the preceding theorem is finite or at most countably infinite and, if infinite, b_1 is the only limit point of E, the set of all end-points of M. By noticing that in the proof of the preceding theorem we make use of four end-points in each step we obtain the following more general result.

THEOREM 10. *A necessary and sufficient condition that an acyclic continuous curve M be homogeneous with respect to n end-points ($n > 3$) is that M contain a point P such that M consists of a set of arcs from P to the end-points of M, no two of which have any point in common except P.*

The following example shows that the condition of Theorem 10 is *not necessary* unless $n > 3$. In this example any three end-points may be carried into any three end-points by a continuous (1,1) transformation of M into itself and yet M fails to satisfy the condition of Theorem 10.

Consider the ordinary Euclidean plane with a rectangular coördinate system. Let M_{11} denote the interval from (1,0) to (0,0) and let M_{12} be vacuous. Let p_i ($i = 1, 2, \cdots$) be the point of M_{11} whose distance from the point (0,0) is $\frac{1}{3} \cdot 2^{1-i}$ and let q_i ($i = 2, 3, 4, \cdots$) be the point of M_{11} whose distance from (1,0) is $\frac{1}{3}(1 + 2^{1-i})$. At each point p_i and q_i erect perpendiculars to M_{11} of length $\frac{1}{15} \cdot 2^{1-i}$ with p_i or q_i, as the case may be, as midpoints. Let M_{21} be the set of points consisting of all such perpendiculars and let M_{22} be vacuous. Let M_2 denote $M_{11} + M_{21}$. Suppose that AB is any interval of M_2 such that A is an end-point of M_2, B is a branch point of M_2 and $\overset{\frown}{AB}$ contains no branch point of M_2. Let

d denote the length of AB. Let u_i $(i = 1, 2, 3, \cdots)$ be the point of AB whose distance from B is $\frac{1}{3}\cdot 2^{1-i}\cdot d$ and let v_i $(i = 2, 3, 4, \cdots)$ be the point of AB whose distance from A is $\frac{1}{3}(1+2^{1-i})d$. At each point u_i and v_i erect perpendiculars to AB of length $\frac{1}{15}\cdot 2^{1-i}\cdot d$ with u_i and v_i, as the case may be, as midpoints. Let M_{31} be the set of points consisting of all such perpendiculars for all such intervals AB of M_2. Let CD be any interval of M_2 such that C and D are branch points of M_2 and $\overset{\frown}{CD}$ contains no branch points of M_2; and let k be the length of CD. Let a_i $(i = 1, 2, 3, \cdots)$ be the point of CD whose distance from C is $2^{-i}\cdot k$ and b_i $(i = 2, 3, 4, \cdots)$ be the point of CD whose distance from D is $2^{-i}\cdot k$. At each point a_i and b_i erect perpendiculars to CD of length $\frac{1}{5}\cdot 2^{-i}\cdot k$ with a_i or b_i, as the case may be, as midpoints. Let M_{32} be the set of points consisting of all such perpendiculars for all such intervals CD of M_2. Let $M_3 = \sum_{i=1}^{3}\sum_{j=1}^{2} M_{ij}$. By erecting similar perpendiculars to all intervals AB and CD of M_3 we may define M_{41} and M_{42}, and so on indefinitely. Let $K_1 = \sum_{i=1}^{\infty}\sum_{j=1}^{2} M_{ij}$ and let K_2 be the set obtained by reflecting K_1 in the line $x = 0$, K_3 be the set obtained by reflecting $K_1 + K_2$ in the line $x = y$. Let $M = K_1 + K_2 + K_3$.

By examination we see that M is an acyclic continuous curve and that any three end-points of M may be carried into any three end-points of M by a continuous (1,1) transformation of M into itself. But M does not satisfy the condition of Theorem 10.

THEOREM 11. *The set of branch points of an acyclic continuous curve is countable.**

LEMMA A. If T is a tree of a continuous curve and P its root then $T+P$ is a continuous curve.

Proof. Let M be the continuous curve. From the definition of a tree it follows that T and $M-(T+P)$ are mutually separated sets. Thus $T+P$ is bounded, closed and connected and we need only to show that $T+P$ is connected im kleinen at each of its points. As $M-T$ is closed, the connectivity im kleinen of $T+P$ at a point Q of T follows at once from the fact that the set M is connected im kleinen at Q. The set $T+P$ is also connected im kleinen at P for if C_1 is any circle with center at P, there exists a concentric circle C_2 such that every point of M lying within

* This theorem was presented to the Society May 1, 1926. I later found a proof by Menger and, as mine is very similar, it is omitted here the reader being referred to Menger's paper (*Über reguläre Baumkurven,* Math. Ann., vol. 97 (1926), pp. 572–582). Since this paper was in print, I have found a still earlier proof by Wazewski, Ann. Soc. Polonaise de Math., vol. 2 (1923), pp. 49–170.

C_2 lies with P in a connected subset of M lying wholly within C_1. Let x be any point of T within C_2 and let M_x be an arc of M from x to P and lying wholly within C_1. Since T and $M-(T+P)$ are mutually separated, the arc M_x must lie wholly in $T+P$ and thus $T+P$ is connected im kleinen at P. Since $T+P$ is bounded, closed, connected and connected im kleinen it is continuous curve.

Definition. Suppose M is a continuous curve which contains just one simple closed curve C. The set $M-C$ consists of a countable number of trees, $d_1, d_2, d_3, \cdots$, by Theorem 1. Each tree d_i has just one limit point P_i in C which is the root of the tree by definition. We shall call the points P_i the *cyclic branch points* of M.

THEOREM 12. *The set of all branch points (ordinary and cyclic) of a continuous curve containing just one simple closed curve is countable.*

Proof. Let M be the continuous curve and C be the simple closed curve in M. Any maximal connected subset of $M-C$ is a tree and the number of such trees is countable by Theorem 1. Let these trees be denoted by $T_1, T_2, T_3, \cdots$ and the corresponding limit points in C by $P_1, P_2, P_3, \cdots$. Every set T_i+P_i is a continuous curve by Lemma A. Also it is evident that no continuous curve T_i+P_i contains a simple closed curve. Then if B_i denotes the set of all the branch points of T_i+P_i, B_i is countable by Theorem 11. Then if B denotes the set of points consisting of all the points of the sets B_i $(i = 1, 2, 3, \cdots)$ together with all the points P_i, B is a countable set being formed of a countable number of countable sets. The set B contains all the branch points of M and therefore the set of all the branch points of M is countable.

IV. On continuous (1,1) correspondences between continuous curves.*

In this section we give a very general set of conditions under which two continuous curves, which are acyclic or each of which contains just one simple closed curve, can be put into continuous (1,1) correspondence. We also give a set of conditions under which a correspondence between subsets of two acyclic continuous curves can be extended to a continuous (1,1) correspondence between the continuous curves themselves.† From this we can formulate the corresponding theorem for the case of the continuous curves containing just one simple closed curve but the conditions are necessarily so numerous as to make the theorem of little interest.

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* Presented to the Society May 1, 1926.

† We say that a correspondence C between two subsets, N_1 and N_2, of two sets, M_1 and M_2, respectively *can be extended* to the sets themselves provided there exists a correspondence C' between M_1 and M_2 such if P is a point of N_1, $C(P) = C'(P)$.

The conditions given are conditions on the subsets, whose correspondence we desire to extend, and the branch points of the continuous curves. Thus we see from Theorems 11 and 12 that the conditions restrict only a countable set of points in addition to the subsets themselves.

THEOREM 13. *Let M_1 and M_2 be acyclic continuous curves and C_1 be a* (1,1) *correspondence between N_1 and N_2, subsets of M_1 and M_2 respectively. Then this correspondence can be extended to a continuous* (1,1) *correspondence T between the continuous curves provided there exists a* (1,1) *correspondence C_2 between B_1 and B_2, the branch points of M_1 and M_2, and C_1 and C_2 satisfy the following conditions:*

(1) *if P is a point of N_1 and B_1 then $C_1(P) = C_2(P)$ and if P' is a point of N_2 and B_2 then $C_1^{-1}(P') = C_2^{-1}(P')$,*

(2) *if P is a point of B_1, P and $C_2(P)$ are branch points of the same order,*

(3) *if P is a point of N_1, P is an ordinary cut-point if, and only if, $C_1(P)$ is an ordinary cut-point,*

(4) *if P_1, P_2, P_3 are points of $K_1(K_i = N_i + B_i)$ then P_1, P_2, P_3 lie on an arc of M_1 and P_2 is between P_1 and P_3 if, and only if, $\bar{C}(P_1)$, $\bar{C}(P_2)$, $\bar{C}(P_3)$ ($\bar{C}$ is the correspondence between K_1 and K_2 which is the same as C_1 for points of N_i and the same as C_2 for points of B_i) lie on an arc of M_2 and $\bar{C}(P_2)$ is between $\bar{C}(P_1)$ and $\bar{C}(P_3)$,*

(5) *a sequence of points of K_1, P_1, P_2, P_3, $\cdots$, has a sequential limit point Q if, and only if, $\bar{C}(P_1)$, $\bar{C}(P_2)$, $\bar{C}(P_3)$, $\cdots$ has a sequential limit point R,*

(6) *if either Q or R is an ordinary cut-point then the other is also,*

(7) *the point Q belongs to N_1 if, and only if, R belongs to N_2.*

It will be noticed that M_1 and M_2 enter into the hypothesis of the theorem in a completely symmetrical manner. Thus any statement which is proved concerning M_1 or any of its subsets gives a corresponding statement concerning M_2 or corresponding subsets of M_2. This being the case we will usually omit the statement concerning M_2. Before proceeding to the proof of the theorem we will state several lemmas which follow out of the hypothesis of the theorem.

LEMMA B. *Under the hypothesis of Theorem* 13, *if Q is a point of K_1 then R is a point of K_2 and $\bar{C}(Q) = R$.*

Proof. We will show that the assumption $\bar{C}(Q) = Q' \neq R$ leads to a contradiction, thus proving the lemma, for $Q' = R$ implies R is a point of K_2. If Q belongs to N_1, R belongs to N_2 by condition (7). If Q belongs to B_1, by condition (6) R belongs to B_2 or is an end-point of M_2. Thus R either belongs to K_2 or is an end-point of M_2. We may suppose that the points Q, P_1, P_2, P_3, $\cdots$ are distinct points as the other cases may be easily settled or reduced to this. Then the points R, P_1', P_2', P_3', $\cdots$,

($P_i' = \bar{C}(P_i)$) are distinct. For every i let b_i' be the first point which the arc of M_2 from P_i' to R has in common with the arc RQ' of M_2. If R is an end-point no b_i' can be R.* If R is a point of K_2, then there exists a point $\bar{R}$ in M_1 such that $\bar{C}(\bar{R}) = R$ and $\bar{R}$ is distinct from Q, If for infinitely many values of i, $b_i' = R$, then for every such i we have the order $P_i'RQ'$ in M_2 and consequently (Condition (4)), $P_i\bar{R}Q$ in M_1 and as M_1 is connected im kleinen Q could not be the sequential limit point of $[P_i]$. Thus in either case not more than a finite number of the points b_i' are R. It follows that there exists a sequence of positive integers, $n_1, n_2, n_3, \cdots (n_i < n_{i+1})$, such that every point b_{n_i}' lies on $\underset{\frown}{RQ'}$, $b_{n_i}' \neq b_{n_j}'$ unless $i = j$ and we have the order $Q'b_{n_1}'b_{n_2}'b_{n_3}' \cdots R$. For every i, b_{n_1}' lies between Q' and P_{n_i}'. The point b_{n_1}' is a point of B_2 and thus a point b_{n_1}, such that $\bar{C}(b_{n_1}) = b_{n_1}'$, exists and lies between Q and P_{n_i} for every i (Condition (4)). As M_1 is connected im kleinen Q cannot be the sequential limit point of the sequence $[P_{n_i}]$. Thus the assumption that $Q' \neq R$ has led to a contradiction.

LEMMA C. *Under the hypothesis of Theorem* 13, *Q is a branch point, an end-point or an ordinary cut-point if, and only if, R is also.*

Proof. By condition (6) if either Q or R is an ordinary cut-point, the other is also. Then the only possible case contradictory to the lemma is that either Q or R, say Q, may be a branch while the other is an end-point. But in this case, by Lemma B, R is a point of K_2 and $Q' = R$. As Q' is a branch point and $Q' = R$, the point R must be a branch point also.

LEMMA D. *Under the hypothesis of Theorem* 13, *if P is a point of N_1 then P and $C_1(P)$ are either both end-points, both branch points or both ordinary cut-points.*

The lemma follows immediately from conditions (1) and (3) of the hypothesis of the theorem.

Proof of Theorem 13. If P is any point of K_1 we denote $\bar{C}(P)$ by P'. We will define T as follows:

If P is any point of K_1, then $T(P) = P'$. If P is not a point of K_1 but is a limit point of K_1, then K_1 contains a sequence, $P_1, P_2, P_3, \cdots$, of points which has P as its sequential limit point. In this case we define $T(P)$ to be the sequential limit point of $P_1', P_2', P_3', \cdots$ which exists by condition (5) and, by Lemma B, $T(P)$ is not a point of K_2.

The transformation T has been defined for all points and limit points of K_1, a closed set U. Let P be any other point of M and $E_{1P}E_{2P}$ be a maximal arc of M_1 containing P. Let p_1 and p_2 be the first points of

* Cf. Theorem 4. Cf. also R. L. Wilder, loc. cit., Theorem 14, and R. L. Moore, *Concerning the cut-points etc.*, loc. cit., Theorem C.

the sets $U+E_{1P}$ and $U+E_{2P}$ on the arcs from P to E_{1P} and from P to E_{2P} respectively. Neither p_1 nor p_2 is P unless P is itself an end-point of M_1. The arc p_1p_2 may be one of three types: (1) $p_1 = E_{1P}$ and $p_2 = E_{2P}$, (2) p_1 and p_2 both belong to U, (3) either p_1 or p_2 is an end-point of M_1 and the other is a point of U.

In case (1) B_1 and B_2 are vacuous and M_1 and M_2 reduce to simple continuous arcs. Here all seven conditions of the hypothesis are satisfied and Theorem 13 is obviously true since any two arcs may be put into continuous (1,1) correspondence.

In case (2) we will show that there is no point of $T(U)$ on the arc of M_2 between $T(p_1)$ and $T(p_2)$. Suppose $T(U)$ has a point Q' between $T(p_1)$ and $T(p_2)$. If Q' is not a point of K_2 it is a limit point of K_2. Let $\varepsilon>0$ be a number smaller than both $d(T(p_1),Q')$ and $d(T(p_2),Q')$. As M_2 is connected im kleinen there exists a number $\delta_\varepsilon>0$ such that every point of M_2 within a distance δ_ε of Q' can be joined to Q' by an arc of M_2 every point of which is a distance from Q' less than ε. As Q' is a limit point of K_2, K_2 contains a point u such that $d(u,Q')<\delta_\varepsilon$. The first point which the arc from u to Q' has in common with the arc from $T(p_1)$ to $T(p_2)$ must be a point of K_2 and lie between $T(p_1)$ and $T(p_2)$. Thus we may assume that Q' is a point of K_2 for, if not, we may replace it by this new point which belongs to K_2.

Let Q denote the point of M_1 such that $\overline{C}(Q) = Q'$; Q exists since Q' is a point of K_2. The first point the arc from Q to p_1 has in common with the arc p_1p_2 is either p_1 or p_2, say p_2. If p_2 is not a point of K_1 then we may replace it in the argument by a point $\overline{p}_2$ of K_1 so that the original order is preserved, as follows:

Let η be a positive number such that the points Q and p_1 lie in the exterior of the circle of radius η with center at p_2 and such that the entire arc of M_2 from Q' to $T(p_1)$ lies in the exterior of the circle of radius η with center at $T(p_2)$. Since M_1 and M_2 are connected im kleinen there exists a positive number δ_η such that every point of $\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$ within a distance δ_η of $\begin{Bmatrix} p_2 \\ T(p_2) \end{Bmatrix}$ can be joined to $\begin{Bmatrix} p_2 \\ T(p_2) \end{Bmatrix}$ by an arc of $\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$ every point of which is within a distance η of $\begin{Bmatrix} p_2 \\ T(p_2) \end{Bmatrix}$. There exists a point q of K_1 such that $d(q,p_2)<\delta_\eta$ and $d(T(q),T(p_2))<\delta_\eta$. Let $\overline{p}_2$ be the first point the arc of M_1 from q to p_2 has in common with the arc Qp_1. Then $\overline{p}_2$ is a point of K_1 and lies between Q and p_1. It follows that $T(\overline{p}_2)$ is a point of K_2 and Q' is between $T(p_1)$ and $T(\overline{p}_2)$. Similarly if p_1 is not a point of K_1 we may determine a point $\overline{p}_1$ of K_1 preserving

order relations. If p_1 or p_2 are points of K_1 let $p_1 = \overline{p}_1$ and $p_2 = \overline{p}_2$; and then in any case we have three points $\overline{p}_1, Q, \overline{p}_2$ of K_1 so that $\overline{p}_2$ is between Q and $\overline{p}_1$ and three corresponding points of K_2 such that Q' is between $\overline{p}_1'$ and $\overline{p}_2'$. But this is contrary to condition (4) and thus the assumption that $T(U)$ contains a point between $T(p_1)$ and $T(p_2)$ has led to a contradiction. Then we may set up any continuous (1,1) correspondence between the arcs p_1p_2 and $T(p_1)T(p_2)$, such that p_1 and $T(p_1)$ and p_2 and $T(p_2)$ correspond, and call this correspondence T for the points of these arcs.

In case (3) we will suppose that p_1 is the end-point of M_1 and p_2 is the point of U. We will consider three possibilities according as p_2 is (a) an end-point, (b) an ordinary cut-point, or (c) a branch point.

In (a), M_1 and M_2 reduce to simple continuous arcs and N_1 and N_2 to a single end-point each. Theorem 13 is here obvious.

In (b), $M_2 - T(p_2)$ is the sum of two trees D_1 and D_2. If neither D_1 nor D_2 is an arc minus one of its end-points, each contains a branch point of M_1, b_1' and b_2' respectively, and $T(p_2)$ is between b_1' and b_2'. Let b_1 and b_2 be the points of M_1 such that $C_2(b_1) = b_1'$ and $C_2(b_2) = b_2'$. Then p_2 is not between b_1 and b_2. If p_2 is a point of K_1, then by condition (4) we have a contradiction immediately, and if not, then as we have done in case (2) above we may replace p_2 by a point of K_1 without disturbing the order arrangement and then obtain a contradiction by condition (4). Therefore at least one of the two trees of $M_2 - T(p_2)$ is a simple arc. If both are simple arcs then M_1 and M_2 reduce to simple arcs and N_1 and N_2 each to a single cut-point and Theorem 13 is evidently true. If just one of the two trees of $M_2 - T(p_2)$ is a simple arc minus one of its end-points, let the end-point which the tree contains be $T(p_1)$ and we may set up any continuous (1,1) correspondence between the two arcs p_1p_2 and $T(p_1)T(p_2)$, such that p_1 and $T(p_1)$ and p_2 and $T(p_2)$ correspond, and call this correspondence T for these two arcs.

In (c), we proceed as follows: The branch points p_2 and p_2' are of the same order by condition (2). We shall show that $M_1 - p_2$ and $M_2 - p_2'$ have the same number (if it is finite, and if one set is infinite then the other is also) of trees that are simple arcs minus one end-point and contain no point of N_1 or N_2. First, let us suppose that p_2 is of order n. Every tree of $M_1 - p_2$ which is not of the above type contains at least one point of K_1. Let $T_1, T_2, \cdots, T_k$ $(0 \leqq k \leqq n)$ be the trees of $M_1 - p_2$ which are not of this type. We shall show that the number of trees $M_2 - p_2'$ not of this type is greater than or equal to k. For k equal 0 or 1 this is evident. Consider $k > 1$. For each i $(0 < i \leqq k)$, T_i contains a point q_i of K_1. For every i and j $(i \neq j)$, p_2 lies between q_i and q_j. Then p_2' lies between q_i' and q_j' by condition (4) and thus q_i' and q_j' lie in different trees of

$M_2 - p_2'$. But as a tree of $M_2 - p_2'$ which contains a point q_i' cannot be a tree of the above mentioned type, the number of trees of $M_2 - p_2'$ not of this type is greater than or equal to k. Conversely in exactly the same manner we may show that k is greater than or equal to the number of trees of $M_2 - p_2'$ which are not of the above type. Therefore $M_1 - p_2$ and $M_2 - p_2'$ each contain exactly $n - k$ trees that are simple arcs minus one end-point and contain no point of N_1 or N_2. The cases where the order of p_2 is of the form $n\infty$ or $\infty\,\infty$ may be treated in a similar manner. Thus in any case the number of trees of $M_1 - p_2$ and $M_2 - p_2'$ that are simple arcs minus one end-point and contain no point of N_1 or N_2, as the case may be, is the same. We may arbitrarily set up a (1,1) correspondence between the trees of p_2 which are of this type and those of p_2' which are of this type and we may select any continuous (1,1) correspondence between corresponding trees (arcs), such that p_2 and p_2' correspond, and call this T for these arcs.

We have now defined the correspondence T for every point of M_1. We have yet to show that T is a (1,1) correspondence between M_1 and M_2 and that T is continuous.

To every point of M_1 there corresponds under T just one point of M_2. It is easy to see that M_2 cannot contain a point which corresponds to no point of M_1. Now suppose two points P_1 and P_2, of M_1 correspond to a single point P of M_2. If P belongs to K_2, then P_1 and P_2 belong to K_1 and we have a contradiction for C is a (1,1) correspondence and for points of K_1, $T = \bar{C}$. If P is not a point of the set K_2 but is a limit point of it, then P_1 and P_2 do not belong to the set K_2 but are limit points of it. Let $P_{11}, P_{12}, P_{13}, \cdots$ and $P_{21}, P_{22}, P_{23}, \cdots$ be sequences of points of K_1 which have P_1 and P_2 respectively as sequential limit points. By definition of T, P is the sequential limit point of both $P_{11}', P_{12}', P_{13}', \cdots$ and $P_{21}', P_{22}', P_{23}', \cdots$ and hence of $P_{11}', P_{21}', P_{12}', P_{22}', \cdots$. But as the sequence $P_{11}, P_{21}, P_{12}, P_{22}, \cdots$ has no sequential limit point this is a contradiction of condition (5). If P is not a point of $T(U)$, then neither P_1 nor P_2 are points of U. As above we define the points p_1 and p_2 such that P, but no point of $T(U)$, is between p_1 and p_2 and p_1 and p_2 are points of $T(U)$ or end-points of M_2. Correspondingly we obtain points $p_{11}, p_{12}, p_{21}, p_{22}$ which are points of U or end-points of M_1 such that P_1, but no point of U, is between p_{11} and p_{12}, and P_2, but no point of U, is between p_{21} and p_{22}, $T(p_{11}) = T(p_{21}) = p_1$, $T(p_{12}) = T(p_{22}) = p_2$. Then either $p_{11} = p_{21}$ and $p_{12} = p_{22}$ and thus there are two arcs of M_1 from p_{11} to p_{12} which is impossible since M_1 is acyclic, or two points of U correspond to one point of $T(U)$ which we have already shown impossible. Therefore T is a (1, 1) correspondence between M_1 and M_2.

We shall show that if P is a limit point of L, a subset of M_1, then $T(P)$ is a limit point of $T(L)$. The converse follows from the symmetry. Let $P_1, P_2, P_3, \cdots$ be a sequence of distinct points of L which has P as its sequential limit point. If infinitely many of the points of the sequence, $P_{k_1}, P_{k_2}, P_{k_3}, \cdots$, belong to K_1 then P''* is the sequential limit point of the sequence, $P''_{k_1}, P''_{k_2}, P''_{k_3}, \cdots$, by Lemma B or definition of T. In this case P'' is a limit point of L''. Suppose infinitely many, $P_{n_1}, P_{n_2}, P_{n_3}, \cdots$, of the points of the sequence are points of U but not of K_1. For each i, there exists a point Q_{n_i} of K_1 such that $d(Q_{n_i}, P_{n_i}) < 2^{-i}$ and $d(Q''_{n_i}, P''_{n_i}) < 2^{-i}$. The point P is the sequential limit point of $[P_{n_i}]$ and consequently of $[Q_{n_i}]$. By condition (5), the sequence $[Q''_{n_i}]$ has a sequential limit point and, by definition of T and Lemma B, this point is P''. Since $d(Q''_{n_i}, P''_{n_i}) < 2^{-i}$, P'' is the sequential limit point of the sequence $[P''_{n_i}]$. In this case then P'' is a limit point of the set L''.

If neither of the above two conditions exist, infinitely many, $P_{m_1}, P_{m_2}, P_{m_3}, \cdots$, of the points of the sequence $[P_i]$ must belong to $M_1 - U$. Each such point lies on an arc $p_{m_i 1} P_{m_i} p_{m_i 2}$ and each point P''_{m_i} on an arc $p''_{m_i 1} P''_{m_i} p''_{m_i 2}$ as these arcs are defined above in the definition of T. One of the two points, $p_{m_i 1}, p_{m_i 2}$, is either a point or a limit point of K_1 and therefore for every i there exists in K_1 a point X_{m_i} such that

$$d(X_{m_i}, P_{m_i}) < d_i,$$

where $d_i = 2^{-i} + d(\text{arc } p_{m_i 1} p_{m_i 2})$, and such that for the corresponding point X''_{m_i} it is true that

$$d(X''_{m_i}, P''_{m_i}) < d'_i,$$

where $d'_i = 2^{-i} + d(\text{arc } p''_{m_i 1} p''_{m_i 2})$. But

$$\lim_{i \to \infty} d_i = 0 \text{ and } \lim_{i \to \infty} d'_i = 0,\dagger$$

therefore as P is the sequential limit point of the sequence $[P_{m_i}]$ it is the sequential limit point of the sequence $[X_{m_i}]$. By condition (5), the sequence $[X''_{m_i}]$ has a sequential limit point which is P'' by definition of T and Lemma B. Then, since $\lim_{i \to \infty} d'_i = 0$, P'' is the sequential limit point of the sequence $[P''_{m_i}]$. Thus in any case if P is a limit point of L, a subset of M_1, P'' is a limit point of L''.

We have defined a correspondence T between M_1 and M_2 such that for any point P of N_1, $T(P) = C_1(P)$, and have shown that T is a (1, 1)

* If P is a point or set of points of M_1, P'' denotes the point or set of points $T(P)$.

† Cf. my paper, Bull. Amer. Math. Soc., vol. 33 (1927), p. 201-208, Theorem 4. Cf. also R. L. Wilder, loc. cit., Theorem 8.

correspondence and is continuous, which completes the proof of the theorem.

Independence examples. Condition (1) of Theorem 13 is simply that the two correspondences C_1 and C_2 be consistent. The other conditions are independently necessary as shown by the following examples. In each example every condition except the one mentioned is satisfied and it is obvious that no continuous (1, 1) correspondence between M_1 and M_2 exists. In each example except (3) and (7), the sets N_1 and N_2 are considered vacuous. We take the continuous curve M_1 in a Euclidean xy-plane and M_2 in a Euclidean $x'y'$-plane.

Condition (2). The set M_1 consists of the points of the interval from $(1, 0)$ to $(-1, 0)$ together with those of the interval from $(0, 1)$ to $(0, 0)$. Let M_2 be the interval (in the $x'y'$-plane) from $(1, 0)$ to $(-1, 0)$ together with the interval from (0,1) to $(0, -1)$. Also to show the necessity of distinguishing various types of branch points of infinite order we give the example: Let L_1 consist of the intervals from $(0, 0)$ to $(-2, 0)$ and from $(-1, \frac{1}{2})$ to $(-1, 0)$. Let L_{i+1} be the set obtained from L_i by shrinking L_i to one-half its size about the point $(0, 0) = P$ and then rotating it about P through an angle of $2^{-i}\frac{\pi}{2}$ in a clockwise direction. Let $M_1 = \sum_{i=1}^{\infty} L_i$, and let M_2 be the same set of points in the $x'y'$-plane as M_1 in the xy-plane with the addition of the interval from (1, 0) to (0, 0). The point P is a branch point of order $0\,\infty$ while the point $P'(= (0, 0)$ in the $x'y'$-plane) is of order $1\,\infty$.

Condition (3). Let M_1 be the interval from $(1, 0)$ to $(-1, 0)$ and N_1 be the point $(0, 0)$. Let M_2 be the interval from $(1, 0)$ to $(-1, 0)$ in the $x'y'$-plane and N_2 be the point $(1, 0)$.

Condition (4). Let M_1 consist of the intervals $(2, 0)$ to $(-2, 0)$, $(1, 1)$ to $(1, -1)$, $(-1, 1)$ to $(-1, -1)$ and $(0, 1)$ to $(0, 0)$. Let M_2 consist of the intervals in the $x'y'$-plane from $(2, 0)$ to $(-2, 0)$, $(1, 1)$ to $(1, 0)$, $(0, 1)$ to $(0, -1)$ and $(-1, 1)$ to $(-1, -1)$.

Condition (5). Let M_1 consist of the intervals $(2, 0)$ to $(-2, 0)$ and, for every positive integer i, from $(1/i, 1/i)$ to $(1/i, 0)$ and from $(-1/i, 1/i)$ to $(-1/i, 0)$. Let M_2 consist of the intervals in its plane from $(3, 0)$ to $(-3, 0)$ and, for every positive integer i, from $(1+1/i, 1/i)$ to $(1+1/i, 0)$ and from $(-1-1/i, 1/i)$ to $(-1-1/i, 0)$.

Condition (6). (a) Let M_1 be the sum of the intervals from $(0, 0)$ to $(2, 0)$ and, for every positive i, from $(1/i, 1/i)$ to $(1/i, 0)$. In this case $Q(= (0, 0))$ is an end-point. Let M_2 be the same as M_1 in its plane with the addition of the interval from $(0, 0)$ to $(-1, 0)$. In M_2 the point $R(= (0, 0))$ is an ordinary cut-point.

(b) Let M_1 be the same set of points in its plane as M_2 in (a) except with the addition of the interval from (0, 1) to (0, 0). In M_1 the point $Q = (0, 0)$ is a branch point. Let M_2 be the same as M_2 in (a) except for the addition of the interval from $(-\frac{1}{2}, 1)$ to $(-\frac{1}{2}, 0)$. In M_2 the point $R = (0, 0)$ is an ordinary cut-point.

Condition (7). Let M_1 be the same set of points in its plane as M_2 in 6(a). Let N_1 be the point (0, 0). Let M_2 be the same as the M_2 of 6(a). Let N_2 be the point $(-\frac{1}{4}, 0)$.

As a corollary of Theorem 13, we have, in case N_1 and N_2 are vacuous:

THEOREM 14. *There exists a continuous* (1, 1) *correspondence between two acyclic continuous curves M_1 and M_2 providing there exists a* (1, 1) *correspondence C between B_1 and B_2, the branch points of M_1 and M_2, satisfying the following conditions*:

(1) *P and $C(P)$ are branch points of the same order,*

(2) *three points of B_1, P_1, P_2, P_3, lie on an arc of M_1 and P_2 is between P_1 and P_3 if, and only if, $C(P_1)$, $C(P_2)$, $C(P_3)$ lie on an arc of M_2 and $C(P_2)$ is between $C(P_1)$ and $C(P_3)$,*

(3) *a sequence of points of B_1, P_1, P_2, P_3, $\cdots$, has a sequential limit point Q if, and only if, the sequence, $C(P_1)$, $C(P_2)$, $C(P_3)$, $\cdots$, has a sequential limit point R,*

(4) *if either Q or R is an ordinary cut-point then the other is also.*

Definition. Suppose P and P' are cyclic branch points of two continuous curves M and M' each of which contains just one simple closed curve, C and C'. Let $\sum D_P$ and $\sum D_{P'}$ be the sets of trees of $M - C$ and $M' - C'$ which have P and P' as roots. Then the points P and P' will be said to be *similar* providing there exists a continuous (1, 1) correspondence T between $P + \sum D_P$ and $P' + \sum D_{P'}$ such that $T(P) = P'$.

The sets $P + \sum D_P$ and $P' + \sum D_{P'}$ are acyclic continuous curves* and thus the conditions under which two cyclic branch points are similar are given in Theorem 13 in which the sets N_1 and N_2 are the single points P and P'.

THEOREM 15. *If M_1 and M_2 are continuous curves each of which contains just one simple closed curve, C_1 and C_2, then there exists a continuous* (1, 1) *correspondence T between M_1 and M_2 provided there exists a* (1, 1) *correspondence T' between B_1 and B_2, the cyclic branch points of M_1 and M_2, satisfying the following conditions*:

(1) *P and $T'(P)$ are similar,*

(2) *the points P_1 and P_2 separate P_3 and P_4 on C_1 if, and only if, the points $T'(P_1)$ and $T'(P_2)$ separate $T'(P_3)$ and $T'(P_4)$ on C_2,*

* Every closed and connected subset of a continuous curve containing only a finite number of simple closed curves is a continuous curve. Cf. abstract of my paper, *Continuous curves of a certain type*, Bull. Amer. Math. Soc., vol. 32 (1926), pp. 307–8.

(3) *the sequence* $P_1, P_2, P_3, \cdots$ *of points of* B_1 *has a sequential limit point* Q *if, and only if, the sequence* $T'(P_1), T'(P_2), T'(P_3), \cdots$ *has a sequential limit point* R,

(4) *the point* Q *is a point of* B_1 *if, and only if,* R *is a point of* B_2.

Proof. We shall define T for points of M_1 as follows:

The points of B_1 are countable and may be ordered in a sequence $p_1, p_2, p_3, \cdots$ and let $p_i' = T'(p_i)$. Let $E_i = p_i + \sum D_{p_i}$ and $E_i' = p_i' + \sum D_{p_i'}$ where $\sum D_{p_i}$ and $\sum D_{p_i'}$ denote the sets of trees of $M_1 - C_1$ and $M_2 - C_2$ which have p_i and p_i' as roots. By condition (1) there exists a continuous (1, 1) correspondence T_i between E_i and E_i' such that $T_i(p_i) = p_i'$. For all points of E_i, let $T = T_i$.

Let P be any point of $C_1 - B_1$ which is a limit point of B_1. Then P is the sequential limit point of a sequence of points $b_1, b_2, b_3, \cdots$ of B_1. By condition (3), the sequence $b_1', b_2', b_3', \cdots$ $(b_i' = T'(b_i))$ has a sequential limit point Q. We define $T(P) = Q$. By condition (4), Q is a point of $C_2 - B_2$. Let L denote the closed set of points for which T has been defined and let $L'' = T(L)$.

Let P be any point of $M_1 - L$. Since L is closed there exist, unless B_1 is vacuous, two points N_{1P} and N_{2P} of L such that one, C_P, of the two arcs of C_1 from N_{1P} to N_{2P} contains P but no point of L except N_{1P} and N_{2P}. In case B_1 is vacuous, the curves M_1 and M_2 are simple closed curves and our theorem is evident. If N_{1P} and N_{2P} are not distinct, B_1 consists of a single point and then our theorem is easily proved. Using methods similar to those of Theorem 13 we may prove that at least one of the two arcs from N_{1P}'' to N_{2P}''* contains no point of L'' except N_{1P}'' and N_{2P}''. If both of the two arcs have this property then B_1 and B_2 each consist of just two points and we may easily complete the proof for this case. Excepting this simple case, let $\bar{C}_P$ be the arc of C_2 from N_{1P}'' to N_{2P}'' which contains no point of L'' with the exception of these two points. For points of the arc C_P we define T to be any continuous (1, 1) correspondence between C_P and $\bar{C}_P$ such that N_{1P} and N_{2P} correspond to N_{1P}'' and N_{2P}'' respectively.

We have now defined T for every point of M_1. It is evident from the definition that no two distinct points of M_1 can correspond to the same point of M_2. And using methods similar to those of Theorem 13 we may show that to each point of M_2 there corresponds at least one point of M_1. Thus T is a (1, 1) correspondence between M_1 and M_2.

It remains to prove that T is continuous. Suppose P is a limit point of a set of points N, a subset of M_1. We shall prove that P'' is a limit

* If K is a point or set of points of M_1, then K' denotes $T'(K)$ and K'' denotes $T(K)$.

point of N''. The set N contains a sequence of distinct points which has P as its sequential limit point and by selecting a subsequence of this we obtain a sequence of points, $p_1, p_2, p_3, \cdots$, of N which has P as its sequential limit point and such that the sequence $p_1'', p_2'', p_3'', \cdots$ of points of N'' has a sequential limit point Q. If $P'' = Q$, our statement is proved.

If P is a point of $M_1 - C_1$, then all but a finite number of the points of the sequence $[p_i]$ must lie in the same tree of $M_1 - C_1$ as P and, since T is continuous for any single tree by definition, $P'' = Q$. If P is a point of $C_1 - L$, all but a finite number of the points of the sequence $[p_i]$ must lie on the interval $N_{1P}N_{2P}$ and, since T is continuous on any such interval, $P'' = Q$. The remaining case is that P is either a point or a limit point of B_1.

If infinitely many of the points p_i lie in trees which have the same root point r, then $r = P$ for $r + \sum D_r$ is closed and has only the point r in common with C_1. Since T is continuous in a single set $r + \sum D_r$, $P'' = Q$. If infinitely many points p_i lie in $M_1 - C_1$ but no single set $r + \sum D_r$ contains more than a finite number of the points, then there exists a sequence of integers $n_1, n_2, n_3, \cdots$ and a sequence of distinct cyclic branch points of M_1, $r_1, r_2, r_3, \cdots$, such that, for every i, $r_i + \sum D_{r_i}$ contains the point p_{n_i}. We see easily that P is the sequential limit point of the sequence $[r_i]$, and that Q is the sequential limit point of the sequence $[r_i']$. Then we may replace the sequences $[p_i]$ and $[p_i'']$ by $[r_i]$ and $[r_i']$ in this case in showing that $P'' \neq Q$ is impossible. Then we need only to consider the case where P is a point or a limit point of B_1 and every point of the sequence $[p_i]$ belongs to C_1. Let $\bar{Q}$ denote the point of M_1 such that $T(\bar{Q}) = Q$, and consider the following cases:

Case I. Infinitely many of the points p_i are points of B_1. Then we may select a subsequence $[p_{k_i}]$ all of whose points belong to B_1. If P is a limit point of B_1 but does not belong to B_1, then P'' was defined as the sequential limit point of a sequence $b_1', b_2', b_3', \cdots$ of B_2 and $P'' \neq Q$ violates condition (3) since the sequence $p_{k_1}, b_1, p_{k_2}, b_2, \cdots$ has the sequential limit point P while the corresponding sequence of B_2 has no sequential limit point. Now suppose P belongs to B_1. By condition (4), Q belongs to B_2 and let $\bar{r}$ be any point of B_1 distinct from P and $\bar{Q}$. Since P is the sequential limit point of the sequence $[p_{k_i}]$ and Q is the sequential limit point of $[p_{k_i}'']$, there exists an integer t_1 such that if $i > t_1$, p_{k_i} lies on $\overset{\frown}{P\bar{Q}}$ or $\overset{\frown}{P\bar{r}}$ and p_{k_i}'' lies on $\overset{\frown}{P'Q}$ or $\overset{\frown}{Q\bar{r}'}$.* If for any value of $i > t_1$, p_{k_i} lies on $\overset{\frown}{P\bar{r}}$ then P and $\bar{r}$ separate $\bar{Q}$ and p_{k_i} on C_1 while P' and $\bar{r}'$

* If A, B, C are points of a simple closed curve γ, AB denotes the arc of γ from A to B which does not contain C. Arcs AC and BC have similar meanings.

do not separate Q and p'_{k_i} on C_2, which contradicts condition (2). The same contradiction arises if for any value of $i > t_1$, p'_{k_i} lies on $\overset{\frown}{Q\bar{r}'}$. Then for every $i > t_1$, p_{k_i} lies on $\overset{\frown}{P\bar{Q}}$ and p'_{k_i} lies on $\overset{\frown}{QP'}$. Let s be an integer greater than t_1. There exists an integer $t_2 > t_1$ such that if $i > t_2$, p_{k_i} lies on the segment of the arc Pp_{k_s} of C_1 which does not contain $\bar{Q}$ or $\bar{r}$. From condition (2), it follows that p'_{k_i} for $i > t_2$ must lie on the segment of the arc $P'p'_{k_i}$ which does not contain Q or $\bar{r}'$. But this arc does not contain Q and so Q cannot be the sequential limit point of the sequence $[p'_{k_i}]$. Thus under the assumption that $P'' \neq Q$, in case infinitely many of the points p_i belong to B_1, we are led to a contradiction.

Case II. *Infinitely many of the points p_i are limit points of B_1 but do not belong to B_1.* Then there exists a subsequence $[p_{k_i}]$ all of whose points are limit points of B_1 but do not belong to B_1. We have just shown that if X is the sequential limit point of a sequence of points of B_1 then X'' is a limit point of the corresponding sequence in B_2. Each point of $[p_{k_i}]$ is a limit point of points of B_1; to each point p_{k_i} we may associate a point b_i of B_1 such that

$$d(p_{k_i}, b_i) < 1/i \text{ and } d(p''_{k_i}, b'_i) < 1/i.$$

As $[p_{k_i}]$ has the sequential limit point P, the point P is the sequential limit point of $[b_i]$. And from the preceding case we have that P'' is a limit point of $[b'_i]$. Since $d(p''_{k_i}, b'_i) < 1/i$, the point P'' is a limit point of $[p''_{k_i}]$. But Q is the only limit point of $[p''_{k_i}]$ and therefore $P'' = Q$.

If neither of the above two cases hold, then all except a finite number of the points p_i belong to C_1 but are neither points nor limit points of B_1. In this case we may show that the assumption $P'' \neq Q$ leads to a contradiction in an analogous manner to that of the similar case of Theorem 13.

Then in every case we have shown that if P is a limit point of a set of points N belonging to M_1, the point P'' is a limit point of N''. The converse follows from the symmetry. Therefore T is a continuous (1, 1) correspondence between M_1 and M_2.

We may set up examples similar to those of Theorem 13 to show that the conditions of this theorem are independently necessary.

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